NOTES PROPORTIONALITY & SIMILARITY

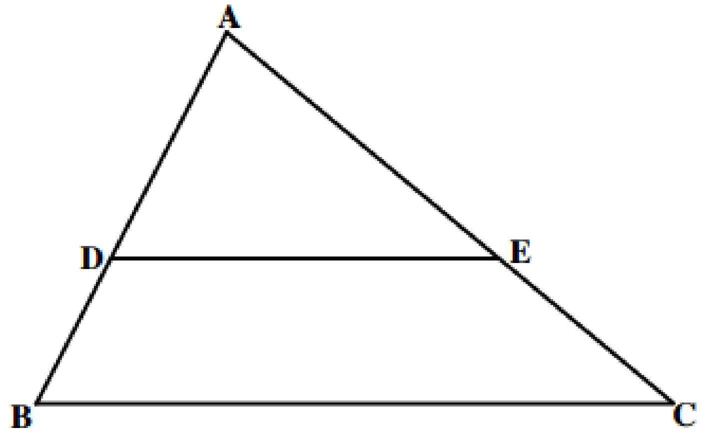
GR 12 PROOFS

THEOREM 1

A line drawn parallel to one side of a triangle cuts the other two sides so as to divide them in the same proportion.

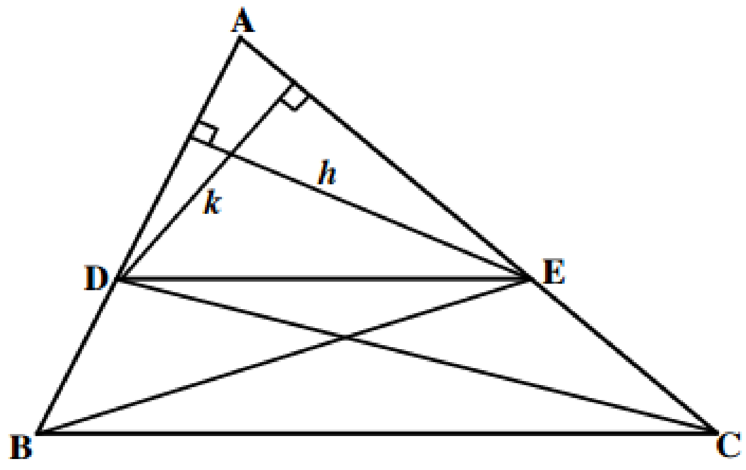
Given: $DE \parallel BC$

Required to prove: $\frac{AD}{DB} = \frac{AE}{EC}$



Proof:

In ΔADE, draw height h relative to base AD and height k relative to base AE.
Join BE and DC to create ΔBDE and ΔCED.



$$\frac{\text{Area } \Delta ADE}{\text{Area } \Delta BDE} = \frac{\frac{1}{2}.AD.h}{\frac{1}{2}.BD.h} = \frac{AD}{BD}$$

$$\frac{\text{Area } \Delta ADE}{\text{Area } \Delta CED} = \frac{\frac{1}{2}.AE.k}{\frac{1}{2}.EC.k} = \frac{AE}{EC}$$

Now it is clear that
Area ΔBDE = Area ΔCED
(same base, height and lying between parallel lines)

$$\therefore \frac{\text{Area } \Delta ADE}{\text{Area } \Delta BDE} = \frac{\text{Area } \Delta ADE}{\text{Area } \Delta CED}$$

$$\therefore \frac{AD}{BD} = \frac{AE}{EC}$$

THEOREM

THEOREM 3

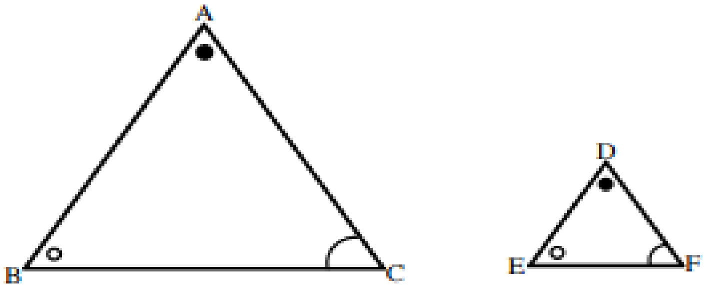
The corresponding sides of two equiangular triangles are in the same proportion and therefore the triangles are similar.

Given:

$\hat{A} = \hat{D}$, $\hat{B} = \hat{E}$ and $\hat{C} = \hat{F}$

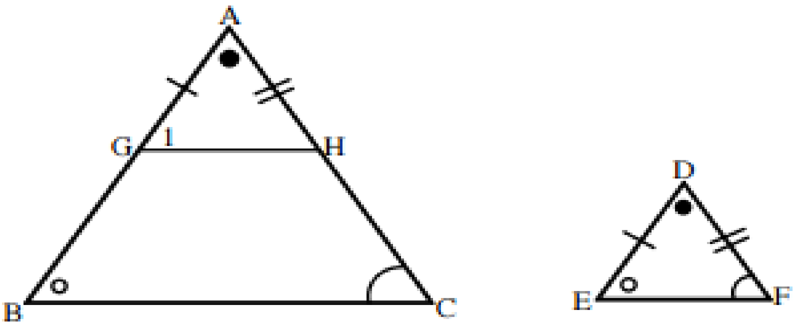
Required to prove:

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$



Proof:

On AB mark off $AG = DE$.
On AC mark off $AH = DF$.
Join GH.
In ΔAGH and ΔDEF:
(1) $AG = DE$ construction
(2) $\hat{A} = \hat{D}$ given
(3) $AH = DF$ construction
$\therefore \Delta AGH \equiv \Delta DEF$ SAS
$\therefore \hat{G}_1 = \hat{E}$
But $\hat{B} = \hat{E}$ given
$\therefore \hat{G}_1 = \hat{B}$
$\therefore GH \parallel BC$ corresponding angles equal.

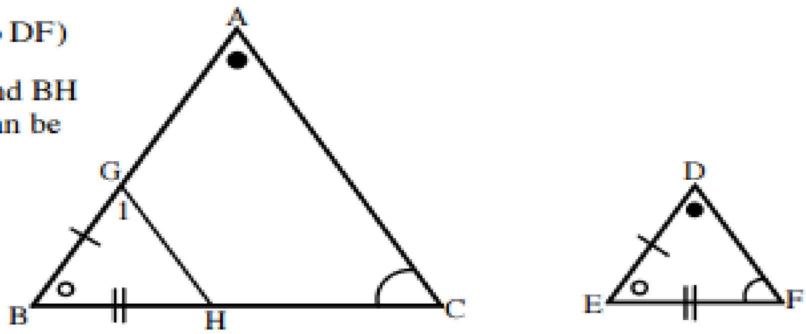


$$\therefore \frac{AB}{AG} = \frac{AC}{AH}$$

$$\therefore \frac{AB}{DE} = \frac{AC}{DF} \quad (AG = DE, AH = DF)$$

Similarly, by constructing BG and BH on AB and BC respectively, it can be proved that

$$\frac{AB}{DE} = \frac{BC}{EF}$$



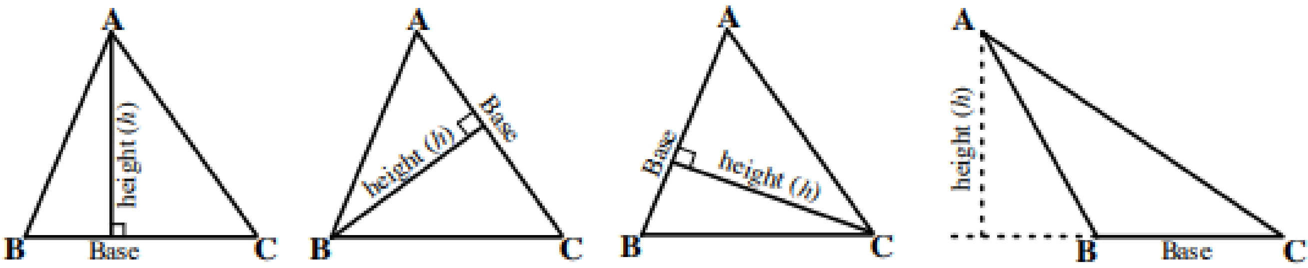
$$\therefore \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

Therefore the triangles are similar.

IMPORTANT CONCEPTS REQUIRED FOR TRIANGLE GEOMETRY

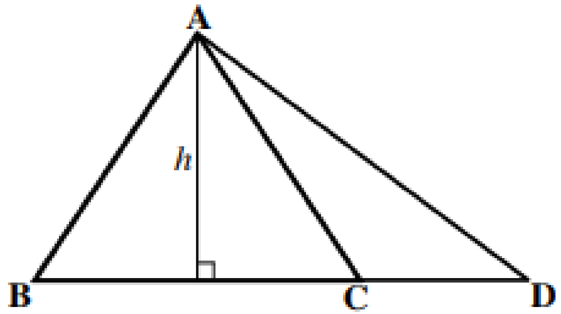
1. Area of Triangles

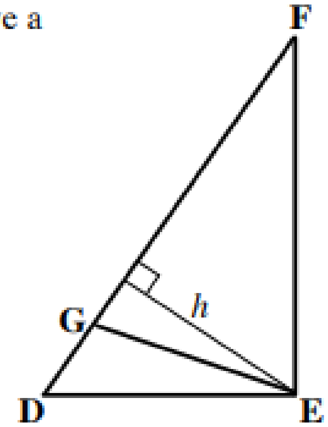
(a) The height or altitude of a triangle is always relative to the chosen base.



In all cases, the area of the triangles can be calculated by using the formula Area $\Delta ABC = \frac{1}{2}(\text{base})(\text{height})$.

(b) Two triangles which share a common vertex have a common height.





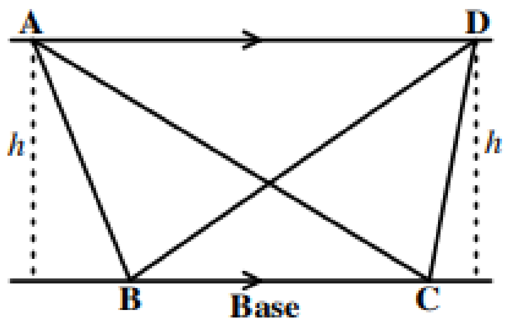
Area $\Delta ABC = \frac{1}{2} . BC . h$

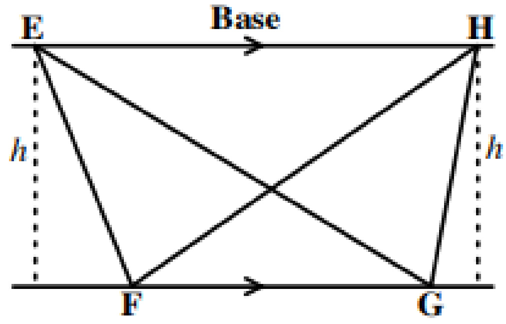
Area $\Delta ACD = \frac{1}{2} . CD . h$

Area $\Delta DEG = \frac{1}{2} . DG . h$

Area $\Delta FGE = \frac{1}{2} . GF . h$

(c) Triangles with equal or common bases lying between parallel lines have the same area.





Area ΔABC = Area ΔDBC

Area ΔEFH = Area ΔHGE

3. **Cross multiplication techniques**

Cross multiplication is useful when working with ratios.

Consider, for example, the ratios $\frac{3}{2} = \frac{6}{4}$

The following statements are true for the given ratio:

If $\frac{3}{2} = \frac{6}{4}$ then:

(a) $\frac{2}{3} = \frac{4}{6}$ (invert both left and right)

(b) $\frac{3}{6} = \frac{2}{4}$ (interchange 6 and 2)

(c) $\frac{4}{2} = \frac{6}{3}$ (interchange 3 and 4)

(d) $3 \times 4 = 6 \times 2$ (multiply 3 by 4 and 6 by 2)

(e) $\frac{3 \times 4}{2} = 6$ (multiply 3 by 4 only)

(f) $3 = \frac{6 \times 2}{4}$ (multiply 6 by 2 only)

In general, for the ratios $\frac{a}{b} = \frac{c}{d}$

If $\frac{a}{b} = \frac{c}{d}$ then:

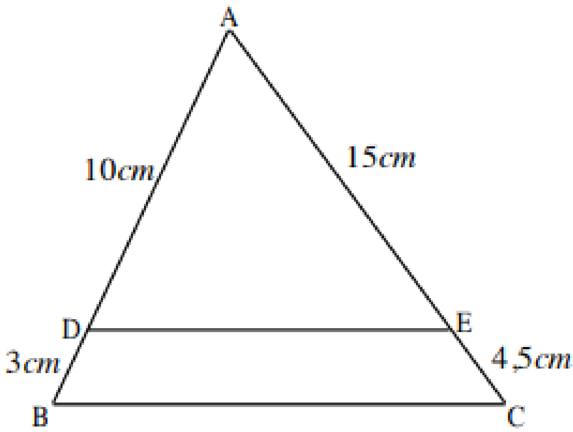
(a) $\frac{b}{a} = \frac{d}{c}$ (b) $\frac{a}{c} = \frac{b}{d}$ (c) $\frac{d}{b} = \frac{c}{a}$

(d) $ad = bc$ (e) $\frac{ad}{b} = c$ (f) $a = \frac{bc}{d}$

EXAMPLE 5

In ΔABC, $AD = 10$ cm, $DB = 3$ cm, $AE = 15$ cm and $EC = 4,5$ cm.
Prove that DE||BC.

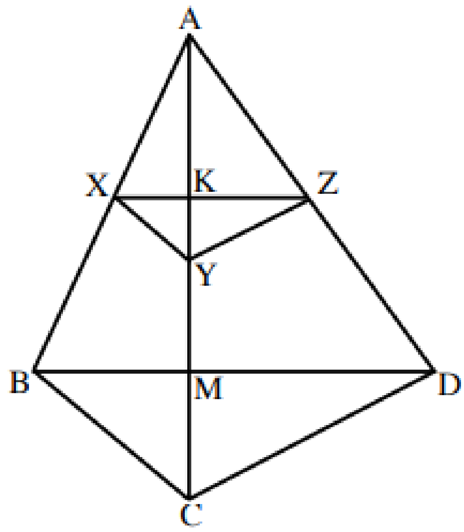
Statement	Reason
$\frac{AD}{DB} = \frac{10cm}{3cm} = \frac{10}{3}$	
$\frac{AE}{EC} = \frac{15cm}{4,5cm}$	
$\therefore \frac{AE}{EC} = \frac{15cm}{\frac{9}{2}cm}$	
$\therefore \frac{AE}{EC} = 15 \times \frac{2}{9}$	
$\therefore \frac{AE}{EC} = \frac{30}{9} = \frac{10}{3}$	
$\therefore \frac{AD}{DB} = \frac{AE}{EC}$	
$\therefore$ DE||BC	Line divides sides of ΔABC prop



EXAMPLE 6

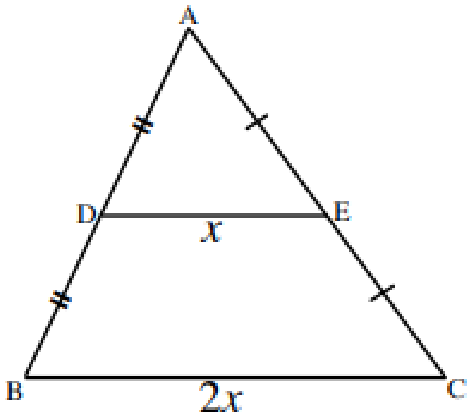
In ΔABC and ΔACD, XY||BC and YZ||CD. Prove that XZ||BD.

Statement	Reason
In ΔABC:	
$\frac{AX}{XB} = \frac{AY}{YC}$	Line || one side of Δ
In ΔACD:	
$\frac{AZ}{ZD} = \frac{AY}{YC}$	Line || one side of Δ
$\therefore \frac{AX}{XB} = \frac{AZ}{ZD}$	Line divides sides of ΔABD prop
$\therefore$ XZ||BD	



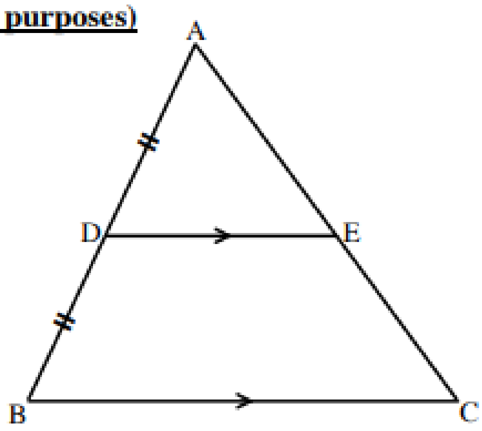
THEOREM 2 (THE MIDPOINT THEOREM)

This theorem is a special case of Theorem 1.
It states that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and equal to half the third side.
If $AD = DB$ and $AE = EC$, then $DE \parallel BC$ and $BC = 2DE$ or $DE = \frac{1}{2}BC$.



THEOREM 2 CONVERSE (not for examination purposes)

The line passing through the midpoint of one side of a triangle, parallel to another side, bisects the third side and is equal to half the length of the side it is parallel to.
If $AD = DB$ and $DE \parallel BC$, then $AE = EC$ and $BC = 2DE$ or $DE = \frac{1}{2}BC$.

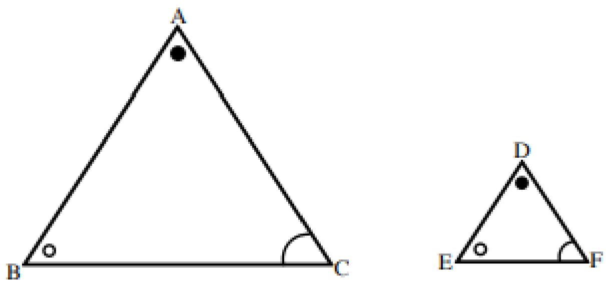


SIMILARITY THEOREMS

Two conditions must **both** be satisfied for two polygons to be similar:

(a) The corresponding angles must be equal.

(b) The ratio of the corresponding sides must be in the same proportion.



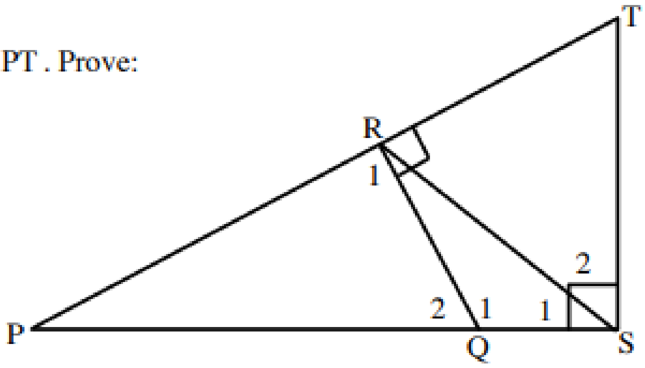
EXAMPLE 8

In ΔPST, $TS \perp PS$ and $RQ \perp PT$. Prove:

(a) $\Delta PRQ ||| \Delta PST$

(b) $RQ : PQ = ST : PT$

(c) $PR \cdot PT = PQ \cdot PS$



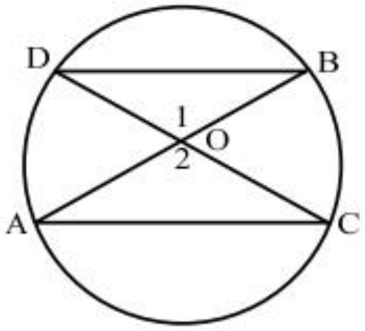
Statement	Reason
(a) Match the corresponding angles of ΔPRQ and ΔPST as follows and then prove the pairs of angles equal.	
$\hat{P}$ —— $\hat{P}$ $\hat{R}_1$ —— $\hat{S}_1 + \hat{S}_2$ $\hat{Q}_2$ ········ $\hat{T}$ Draw solid lines for each pair of corresponding angles that are equal. The dotted line indicates that the pair of angles are equal due to the sum of the angles of a triangle.	
In ΔPRQ and ΔPST:	
(1) $\hat{P} = \hat{P}$	common
(2) $\hat{R}_1 = \hat{S}_1 + \hat{S}_2 = 90°$	given
(3) $\hat{Q}_2 = \hat{T}$	sum of angles of Δ
$\therefore \Delta PRQ ||| \Delta PST$	$\angle, \angle, \angle$
(b) Since $\Delta PRQ ||| \Delta PST$:	
$\frac{PR}{PS} = \frac{RQ}{ST} = \frac{PQ}{PT}$	corr sides of Δ's in proportion
$\therefore \frac{RQ}{ST} = \frac{PQ}{PT}$	
$\therefore \frac{RQ}{PQ} = \frac{ST}{PT}$	cross multiplication
$\therefore RQ:PQ = ST:PT$	
(c) $\frac{PR}{PS} = \frac{PQ}{PT}$	since $\frac{PR}{PS} = \frac{RQ}{ST} = \frac{PQ}{PT}$
$\therefore PR \cdot PT = PQ \cdot PS$	cross multiplication

EXAMPLE 9

A, B, C and D are concyclic points. DOC and AOB are chords. DB and AC are joined. Prove that:

(a) $\Delta AOC \,|||\, \Delta DOB$

(b) $\dfrac{OB}{OD} = \dfrac{OC}{OA}$



	Statement	Reason
(a)	Match the corresponding angles of ΔAOC and ΔDOB as follows and then prove the pairs of angles equal.	
	$\hat{A}$ —— $\hat{D}$ $\hat{O}_2$ ·········· $\hat{O}_1$ $\hat{C}$ —— $\hat{B}$ Draw solid lines for each pair of corresponding angles that are equal. The dotted line indicates that the pair of angles are equal due to the sum of the angles of a triangle.	
	In ΔAOC and ΔDOB:	
	(1) $\hat{A} = \hat{D}$	arc BC subtends equal angles
	(2) $\hat{C} = \hat{B}$	arc AD subtends equal angles
	(3) $\hat{O}_2 = \hat{O}_1$	sum of angles of Δ
	$\therefore \Delta AOC \,|||\, \Delta DOB$	$\angle, \angle, \angle$
	Note: You could have also used the reason "vertically opposite angles" for statement (3) above.	
(b)	$\Delta AOC \,|||\, \Delta DOB$ $\therefore \dfrac{AO}{DO} = \dfrac{OC}{OB} = \dfrac{AC}{DB}$	corr sides of Δ's in proportion
	$\therefore \dfrac{OA}{OD} = \dfrac{OC}{OB}$	
	$\therefore \dfrac{OB}{OD} = \dfrac{OC}{OA}$	cross multiplication