MONYETLA PROJECT SUMMER CAMP DEC 2020

MATHEMATICS

EUCLIDEAN GEOMETRY

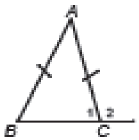
GRADE 11

NOTES

EUCLIDEAN GEOMETRY

CIRCLE GEOMETRY

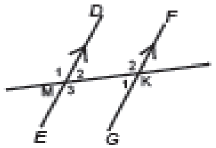
FLASHBACK: Theory from previous grades



$\hat{B} = \hat{C}_1$ (∠'s opp. = sides)

$\hat{A} + \hat{B} + \hat{C}_1 = 180°$ (sum ∠'s of Δ)

$\hat{C}_2 = \hat{A} + \hat{B}$ (ext. ∠'s of Δ)



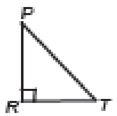
$\hat{K}_2 = \hat{M}_1$ (corres. ∠'s DE//GF)

$\hat{K}_2 = \hat{M}_3$ (alt. ∠'s DE//GF)

$\hat{K}_2 + \hat{M}_2 = 180°$ (co-int. ∠'s DE//GF)

$\hat{M}_1 = \hat{M}_3$ (vert. opp. ∠'s)

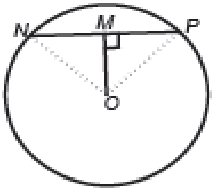
$\hat{K}_2 + \hat{K}_1 = 180°$ (∠'s on a str. line)



$PT^2 = PR^2 + RT^2$ (Pythag. Th.)

Theorem 1: (line from centre ⊥ chord)

A line drawn from the centre of a circle perpendicular to a chord bisects the chord.



GIVEN: Circle centre O with chord $NP \perp MO$.

RTP: $NM = MP$

PROOF:

Join ON and OP

In ΔMON and ΔMOP

$N\hat{M}O = P\hat{M}O$ (OM⊥PN, given)

$ON = OP$ (radii)

$OM = OM$ (common)

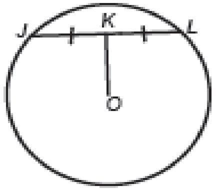
$\therefore \Delta MON \equiv \Delta MOP$ (RHS)

$NM = MP$

Converse of Theorem 1: (line from centre mid-pt. chord)

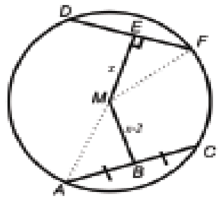
The line segment joining the centre of a circle to the midpoint of a chord is perpendicular to the chord.

If $JK = KL$, then $OK \perp JL$



EXAMPLE

Given circle centre M with a diameter of 20 cm and chord DF of 12 cm.



Determine the length of of chord AC.

Join MF

$DE = EF = 6$ cm (line from centre ⊥ chord)

$MF = 10$ cm (radius)

$x^2 = 10^2 - 6^2$ (Pythag. Th.)

$x^2 = 64$

$x = 8$ cm

$\therefore MB = 8 - 3 = 5$ cm (given)

Join MA

$MA \perp AC$ (line from centre mid-pt. chord0

$MA = 10$ cm (radius)

$AB^2 = 10^2 - 5^2$ (Pythag. Th.)

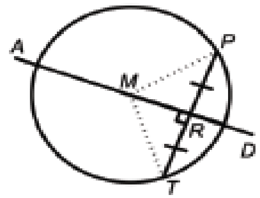
$AB^2 = 75$

$AB = 8{,}66$ cm

$\therefore AC = 17{,}32$ cm

Converse two of Theorem 1: (perp bisector of chord)

The perpendicular bisector of a chord passes through the centre of the circle.



GIVEN: $RT = RP$ and $MR \perp TP$

RTP: MR goes through the centre of the circle.

PROOF:

Choose any point, say M, on AD.

Join MT and MP

In ΔMRP and ΔMRT

$PR = RT$ (given)

$MR = MR$ (common)

$M\hat{R}P = M\hat{R}T = 90°$ (∠'s on a str. line)

$\Delta MRT \equiv \Delta MRP$ (SAS)

$\therefore MT = MP$

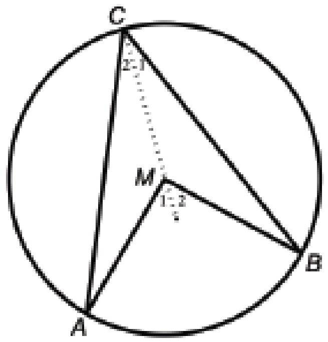
∴All points on AD are equidistant from P and T and the centre is equidistant from P and T.

∴The centre lies on AD.

CIRCLE GEOMETRY

Theorem 2:
(∠ at centre = 2 x ∠ at circum.)

The angle subtended by an arc at the centre of the circle is twice the angle the arc subtends at any point on the circumference of the circle.



GIVEN: Circle centre M with arc AB subtending $A\hat{M}B$ at the centre and $A\hat{C}B$ at the circumference.

RTP: $A\hat{M}B = 2 \times A\hat{C}B$

PROOF:

$AM = BM = CM$ (radii)

$\hat{A} = \hat{C}_2$ (∠'s opp. = sides)

$\hat{B} = \hat{C}_1$ (∠'s opp. = sides)

$\hat{M}_1 = \hat{A} + \hat{C}_2$ (ext. ∠ of Δ)

$\therefore \hat{M}_1 = 2\hat{C}_2$

$\hat{M}_2 = \hat{B} + \hat{C}_1$ (ext. ∠ of Δ)

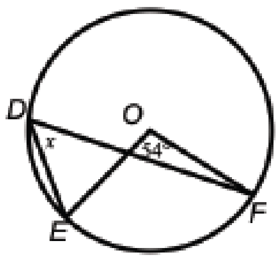
$\therefore \hat{M}_2 = 2\hat{C}_1$

$\therefore \hat{M}_1 + \hat{M}_2 = 2(\hat{C}_1 + \hat{C}_2)$

$\therefore A\hat{M}B = 2 \times A\hat{C}B$

EXAMPLE 1

Determine the value of x:

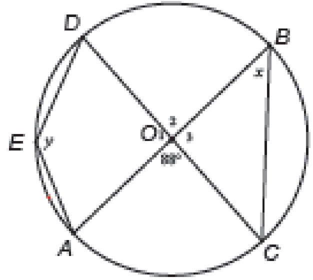


$x = 54° \div 2$ (∠ at centre = 2 x ∠ at circum.)

$\therefore x = 27°$

EXAMPLE 2

Determine the value(s) of x and y:



$x = 44°$ (∠ at centre = 2 x ∠ at circum.)

$OB = OC$ (radii)

$\hat{C} = 44°$ (∠'s opp. = sides)

$\hat{O}_3 = 92°$ (sum ∠'s of Δ)

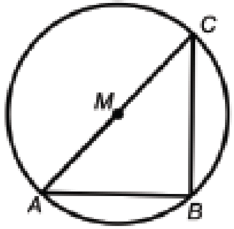
$\hat{O}_2 = 88°$ (vert. opp. ∠'s)

$y = \dfrac{88° + 92° + 88°}{2}$

$y = 137{,}5°$ (∠ at centre = 2 x ∠ at circum.)

Theorem 3:
(∠ in semi-circle)

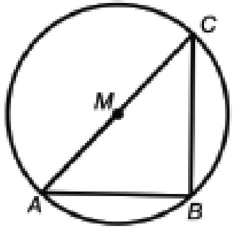
The angle subtended by the diameter at the circumference of a circle is a right angle.



If AMC is the diameter then $\hat{B} = 90°$.

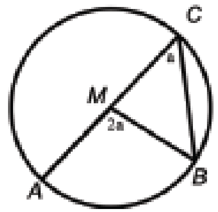
Converse Theorem 3:
(chord subtends 90°)

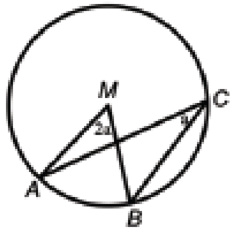
If a chord subtends an angle of 90° at the circumference of a circle, then that chord is a diameter of the circle.

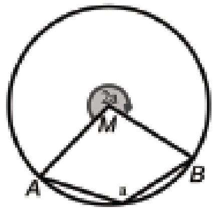


If $\hat{B} = 90°$ then AMC is the diameter.

ALTERNATIVE DIAGRAMS:

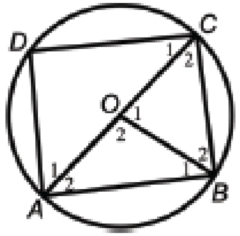






EXAMPLE

In circle O with diameter AC, $DC = AD$ and $\hat{B}_2 = 56°$. Determine the size of $D\hat{A}B$



$CO = OB$ (radii)

$\hat{C}_2 = \hat{B}_2 = 56°$ (∠'s opp. = sides)

$\hat{O}_1 = 68°$ (sum ∠'s of Δ)

$\hat{A}_2 = 34°$ (∠ at centre = 2 x ∠ at circum.)

$\hat{D} = 90°$ (∠ in semi-circle)

$\hat{A}_1 = \hat{C}_1$ (∠'s opp. = sides, DC = AD)

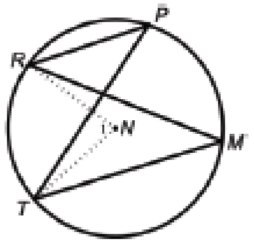
$\hat{A}_1 = 45°$ (sum ∠'s of Δ)

$\therefore D\hat{A}B = 34° + 45° = 79°$

Theorem 4:

(∠ in same seg.)

Angles subtended by a chord (or arc) at the circumference of a circle on the same side of the chord are equal.



GIVEN: Circle centre N with arc RT subtending $R\hat{P}T$ and $R\hat{M}T$ in the same segment.

RTP: $R\hat{P}T = R\hat{M}T$

PROOF:

Join NR and NT to form $\hat{N}_1$.

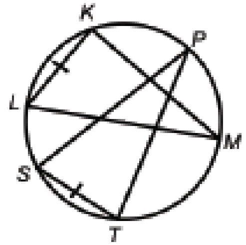
$\hat{M} = \frac{1}{2} \times \hat{N}_1$ (∠ at centre = 2 x ∠ at circum.)

$\hat{P} = \frac{1}{2} \times \hat{N}_1$ (∠ at centre = 2 x ∠ at circum.)

$\therefore R\hat{M}T = R\hat{P}T$

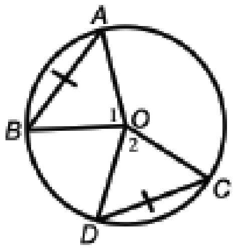
COROLLARIES:

a) Equal chords (or arcs) subtend equal angles at the circumference.



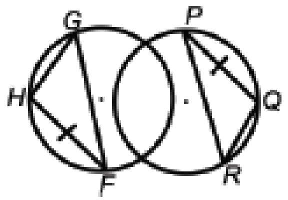
$KL = ST$ then $\hat{P} = \hat{M}$ (= chords, = ∠'s)

b) Equal chords subtend equal angles at centre of the circle.



If $AB = CD$ then $\hat{O}_1 = \hat{O}_2$ (= chords, = ∠'s)

c) Equal chords in equal circles subtend equal angles at their circumference.

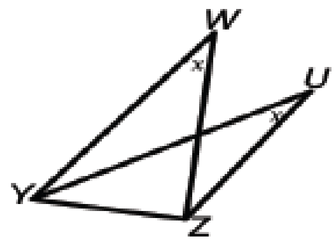


If $HF = PQ$ then $\hat{G} = \hat{R}$ (= chords, = ∠'s)

Converse Theorem 4:

(line subt. = ∠'s)

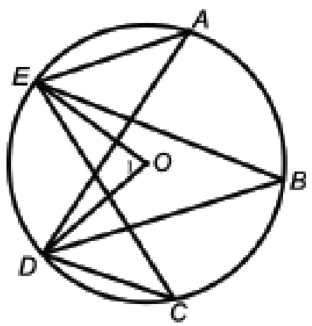
If a line segment joining two points subtends equal angles at two other points on the same side of the line segment, then these four points are concyclic (that is, they lie on the circumference of a circle.)



If $\hat{W} = \hat{U}$, then $WUZY$ is a cyclic quadrilateral.

EXAMPLE 1

Given circle centre O with $\hat{C} = 36°$



Calculate the values of angles:

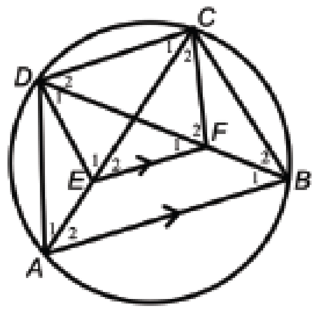
$\hat{O}_1, \hat{A}$ and $\hat{B}$.

$\hat{O}_1 = 2 \times 36° = 72°$ (∠ at centre = 2 x ∠ at circum.)

$\hat{A} = \hat{B} = \hat{C} = 36°$ (∠'s same seg.)

EXAMPLE 2

Given circle $ABCD$ with $AB || EF$.



Questions:

a) Prove $CDEF$ is a cylindrical quad.

b) If $\hat{D}_2 = 38°$, calculate $\hat{E}_2$

Solutions:

a) $\hat{B}_1 = \hat{C}_1$ (∠'s same seg.)

$\hat{B}_1 = \hat{F}_1$ (corres. ∠'s, AB||EF)

$\therefore \hat{C}_1 = \hat{F}_1$

$\therefore CDEF$ cyc. quad (line subt = ∠'s)

b) $\hat{D}_2 = \hat{E}_2 = 38°$ (∠'s same seg quad CDEF)

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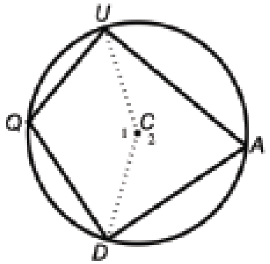
Theorem 5: (opp. ∠'s cyc. quad)

The opposite angles of a cyclic quadrilateral are supplementary.



GIVEN: Circle centre C with quad $QUAD$.

RTP: $\hat{Q} + \hat{A} = 180°$

PROOF:

Join UC and DC

$\hat{C}_1 = 2\hat{A}$ (∠ at centre = 2 x ∠ at circum.)

$\hat{C}_2 = 2\hat{Q}$ (∠ at centre = 2 x ∠ at circum.)

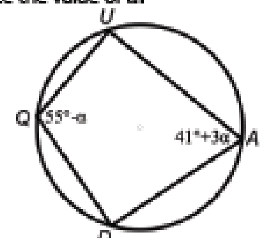
$\hat{C}_1 + \hat{C}_2 = 360°$ (∠'s around a pt.)

$\therefore 2\hat{A} + 2\hat{Q} = 360°$

$\therefore \hat{A} + \hat{Q} = 180°$

EXAMPLE 1

Calculate the value of α.



$55° - \alpha + 41° + 3\alpha = 180°$ (opp. ∠'s cyc. quad)

$2\alpha = 180° - 96°$

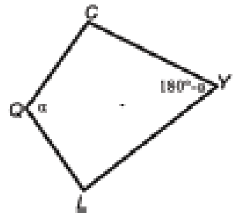
$2\alpha = 84°$

$\therefore \alpha = 42°$

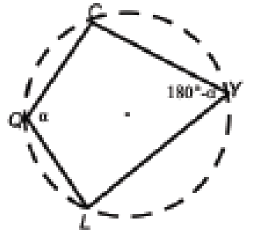
Converse Theorem 5: (opp. ∠'s quad supp)

If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

If $\hat{Q} + \hat{Y} = 180°$
or $\hat{C} + \hat{L} = 180°$

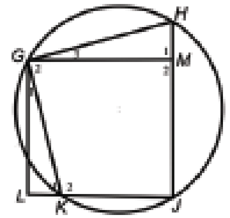


Then $QCYL$ is cyclic



EXAMPLE 2

Given circle $GHJK$ with $GM \perp HJ$ and $GL \perp LJ$. $\hat{G}_3 = 24°$



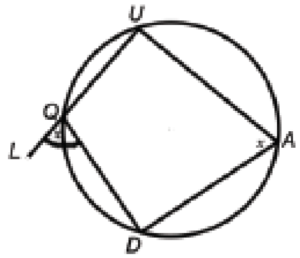
a) Is quadrilateral $GLJM$ a cyclic quad?
b) Is quadrilateral $GLJH$ a cyclic quad?

a) $\hat{M}_2 = 90°$ (Given $GM \perp HJ$)
$\hat{L} = 90°$ (Given $GL \perp LJ$)
$\therefore GLJM$ cyc quad (opp ∠'s quad suppl)

b) $\hat{H} = 180° - 24° - 90°$ (sum ∠'s of Δ)
$\hat{H} = 66°$
$GLJH$ not cyclic (opp ∠'s = 156° not 180°)

Theorem 6: (ext. ∠ cyc quad)

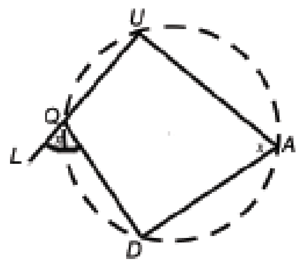
The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.



$L\hat{Q}D = \hat{A}$ (ext. ∠ cyc quad)

Converse Theorem 6: (ext. ∠ = int. opp. ∠)

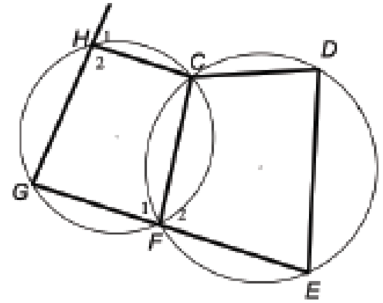
If the exterior angle of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is cyclic.



If $L\hat{Q}D = \hat{A}$ then $QUAD$ is cyclic

EXAMPLE 1

GFE is a double chord and $\hat{H}_1 = 75°$

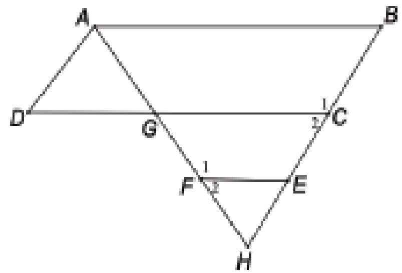


Determine the value of $\hat{D}$.

$\hat{H}_1 = \hat{F}_1 = 75°$ (ext. ∠ cyc quad)
$\hat{F}_1 = \hat{D} = 75°$ (ext. ∠ cyc quad)

EXAMPLE 2

$ABCD$ is a parallelogram and $B\hat{A}D = \hat{F}_1$.
Prove that $CEFG$ is a cyclic quad.



$B\hat{A}D = \hat{C}_1$ (opp. ∠'s parm)
$B\hat{A}D = \hat{F}_1$ (given)
$\therefore \hat{C}_1 = \hat{F}_1$
$\therefore CEFG$ is a cyc quad (ext. ∠ = int. opp. ∠)

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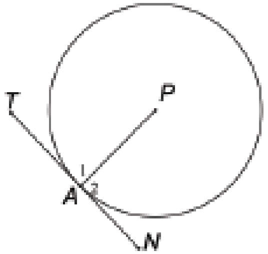
Theorem 7: (tan ⊥ radius)

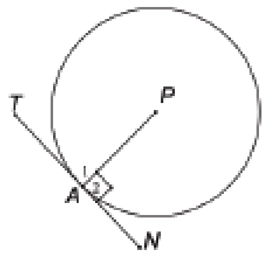
A tangent to a circle is perpendicular to the radius at its point of contact.



If TAN is a tangent to circle P, then $PA \perp TAN$

Converse Theorem 7: (line seg ⊥ radius)

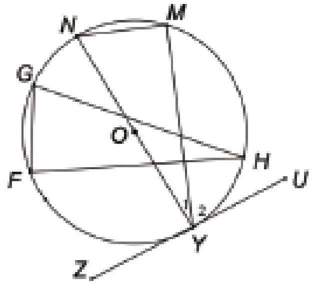
A line drawn perpendicular to the radius at the point where the radius meets the circumference is a tangent to the circle.



If $PA \perp TAN$, then TAN is a tangent to circle P.

EXAMPLE 1

Given circle centre O with tangent ZYU and $MN = FG$. If $\hat{H} = 18°$ determine the size of $\hat{Y}_2$.



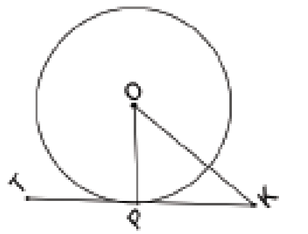
$\hat{Y}_1 = \hat{H} = 18°$ (equal chords, = ∠'s)

$\hat{Y}_1 + \hat{Y}_2 = 90°$ (tan ⊥ radius)

$\therefore \hat{Y}_2 = 90° - 18° = 72°$

EXAMPLE 2

Prove that TPK is a tangent to circle centre O and radius of 8 cm, if $OK = 17$ cm and $PK = 15$ cm.



$OK^2 = 17^2 = 289$

$OP^2 + PK^2 = 8^2 + 15^2$

$= 289$

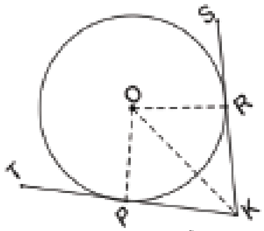
$\therefore OK^2 = OP^2 + PK^2$

$\therefore OP \perp TPK$ (conv. Pythag. Th.)

$\therefore TPK$ is a tan to circle O (line seg ⊥ radius)

Theorem 8: (tan from same pt.)

Two tangents drawn to a circle from the same point outside the circle are equal in length.



GIVEN: Tangents TPK and SRK to circle centre O.

RTP: $PK = RK$

PROOF:

Construct radii OR and OP and join OK.

In ΔOPK and ΔORK

$OP = OR$ (radii)

$OK = OK$ (common)

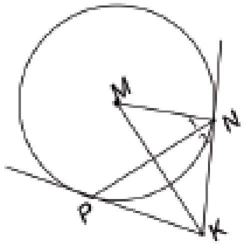
$O\hat{P}K = O\hat{R}K = 90°$ (tan ⊥ radius)

$\therefore \Delta OPK \equiv \Delta ORK$ (RHS)

$\therefore PK = RK$

EXAMPLE

PK and KN are tangents to circle centre M. If $\hat{N}_1 = 24°$, determine the size of $P\hat{K}N$.



$M\hat{N}K = 90°$ (tan ⊥ radius)

$\therefore \hat{N}_2 = 66°$

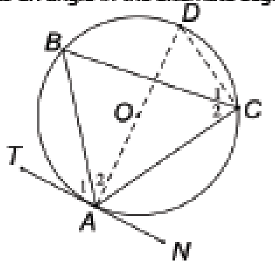
$PK = NK$ (tan from same pt.)

$\hat{N}_2 = N\hat{P}K = 66°$ (∠'s opp. = sides)

$\therefore P\hat{K}N = 48°$ (sum ∠'s of Δ)

Theorem 9: (tan-chord th.)

The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment.



GIVEN: Tangent TAN to circle O, and chord AC subtending $\hat{B}$.

RTP: $\hat{A}_1 = \hat{C}_2$

PROOF:

Draw in diameter AOD an join DC.

$\hat{A}_1 + \hat{A}_2 = 90°$ (tan ⊥ radius)

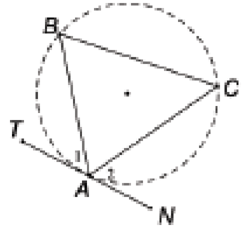
$\hat{C}_1 + \hat{C}_2 = 90°$ (∠ in semi-circle)

$\hat{A}_2 = \hat{C}_1$ (∠'s in same seg)

$\therefore \hat{A}_1 = \hat{C}_2$

Converse Theorem 9: (∠ betw. line and chord)

If a line is drawn through the end point of a chord, making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.



If $\hat{A} = \hat{C}$ or $\hat{A}_2 = \hat{B}$, TAN a tangent

EXAMPLE 1

TRN is a tangent at R and $SR = RQ$. If $\hat{R}_1 = x$, find five angles equal to x.

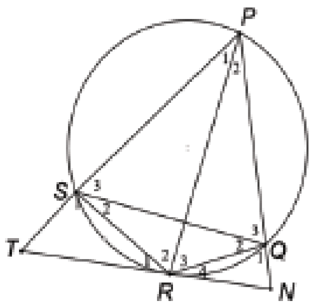
$\hat{R}_1 = \hat{P}_1 = x$ (tan-chord th.)

$\hat{Q}_2 = x$ (tan-chord or ∠'s in same seg)

$\hat{Q}_2 = \hat{S}_2 = x$ (∠'s opp. = sides)

$\hat{S}_2 = \hat{P}_2 = x$ (∠'s same seg)

$\hat{P}_2 = \hat{R}_4 = x$ (tan-chord th.)

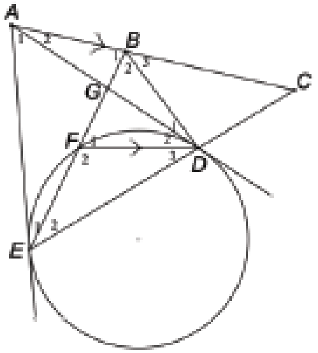


EUCLIDEAN GEOMETRY

CIRCLE GEOMETRY

EXAMPLE 2

In the figure, AD and AE are tangents to the circle DEF. The straight line drawn through A, parallel to FD meets ED produced at C and EF produced at B. The tangent AD cuts EB at G.



a) Prove that $ABDE$ is a cyclic quadrilateral given $\hat{E}_2 = x$.

b) If it is further given that $EF = DF$, prove that ABC is a tangent to the circle passing through the points B, F and D.

a) $\hat{E}_2 = \hat{D}_2 = x$ (tan-chord th.)

$\hat{D}_2 = \hat{A}_2 = x$ (alt ∠'s AB||FD)

$\therefore ABDE$ a cyc quad (line seg subt. = ∠'s)

b) $\hat{E}_2 = \hat{D}_3 = x$ (∠'s opp. = sides)

$\hat{F}_1 = \hat{E}_2 + \hat{D}_3 = 2x$ (ext. ∠ of Δ)

$AE = AD$ (tan from same pt.)

$\hat{E}_1 + \hat{E}_2 = \hat{D}_2 + \hat{D}_3 = 2x$ (∠'s opp. = sides)

$\therefore \hat{B}_3 = 2x$ (ext. ∠ cyc quad)

$\hat{B}_3 = \hat{F}_1$

$\therefore ABC$ tan to circle (∠ betw. line and chord)

ALTERNATIVE

$\hat{F}_1 = \hat{B}_1$ (alt ∠'s AB||FD)

$\hat{B}_1 = \hat{D}_2 + \hat{D}_3$ (∠'s same seg)

$\hat{D}_1 = \hat{E}_1$ (∠'s same seg)

$\hat{E}_1 = \hat{D}_3$ (tan-chord th.)

$\therefore \hat{B}_1 = \hat{D}_2 + \hat{D}_1$

$\therefore ABC$ tan to circle (∠ betw. line and chord)

Hints when answering Geometry Questions

- Read the given information and mark on to the diagram if not already done.
- Never assume anything. If not given or marked on diagram is not true unless proved.
- As you prove angles equal or calculate angles mark them on to the diagram and write down statement and reason there and then.
- Make sure that by the end of the question you have used all the given information.
- If asked to prove something, it is true.
  For EXAMPLE if ask to prove ABCD a cyclic quad, then it is, but if you can't then you can use it as one in the next part of the question.