ANALYTICAL GEOMETRY

What is Analytical Geometry?

Analytical Geometry (Co-ordinate Geometry): Application of straight line functions in conjunction with Euclidean Geometry by using points on a Cartesian Plane.

FLASHBACK

Straight line parallel to the x-axis: m = 0

Straight line parallel to the y-axis: m = undefined

Straight line equation:

$y = mx + c$

Gradient formula:

$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

Parallel gradients:

$m_1 = m_2$

Perpendicular gradients:

$m_1 \times m_2 = -1$

Distance:

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Co-linear:

$m_{AB} = m_{BC}$ OR $d_{AB} + d_{BC} = d_{AC}$

Collinear points A, B and C lie on the same line

Midpoint formula:

$M(x; y) = \left(\frac{x_2 + x_1}{2}; \frac{y_2 + y_1}{2}\right)$

Midpoint Theorem: If two midpoints on adjacent sides of a triangle are joined by a straight line, the line will be parallel to and half the distance of the third side of the triangle.

EXAMPLE

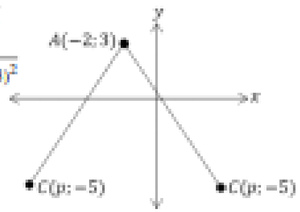
Given: $A(-2;3)$ and $C(p;-5)$ are points on a Cartesian Plane.

1. If $AC = 10$ units determine the value(s) of p.
2. If $C(4;-5)$, determine the equation of the line AC.
3. Determine the co-ordinates of M, the midpoint of AC.
4. If $B\left(-1;\frac{5}{3}\right)$ determine if A, B and C are collinear.
5. Determine the equation of the line perpendicular to AC passing through B.

SOLUTION

1. Draw a sketch diagram. C has two potential x-coordinates for p.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$10 = \sqrt{(p-(-2))^2 + (-5-3)^2}$$
$$100 = (p+2)^2 + 64$$
$$100 = p^2 + 4p + 4 + 64$$
$$0 = p^2 + 4p - 32$$
$$0 = (p+4)(p+8)$$
$$p = 4 \text{ or } p = -8$$



2. **Line equation** requires solving m and c.

$$m = \frac{\Delta y}{\Delta x}$$
$$m_{AC} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{3-(-5)}{-2-4}$$
$$= -\frac{4}{3}$$

$$y = mx + c$$
$$(3) = -\frac{4}{3}(-2) + c$$
$$c = \frac{1}{3}$$

$$\therefore y = -\frac{4}{3}x + \frac{1}{3}$$

3. **Midpoint formula**

$$M(x;y) = \left(\frac{x_2 + x_1}{2}; \frac{y_2 + y_1}{2}\right)$$
$$= \left(\frac{-2+4}{2}; \frac{3+(-5)}{2}\right)$$
$$M(1;-1)$$

4. **Prove collinearity** by proving that the points share a common gradient.

$$m = \frac{\Delta y}{\Delta x} \qquad m = \frac{\Delta y}{\Delta x}$$
$$m_{AB} = \frac{3-\frac{5}{3}}{-2-(-1)} \qquad m_{BC} = \frac{\frac{5}{3}-(-5)}{-1-4}$$
$$m_{AB} = -\frac{4}{3} \qquad m_{BC} = -\frac{4}{3}$$

$\therefore A$, B and C are collinear

5. **Line equation** requires solving m_2 and c w.r.t. B.

$$m_{AC} \times m_2 = -1$$
$$-\frac{4}{3} \times m_2 = -1$$
$$m_2 = \frac{3}{4}$$

$$y = mx + c$$
$$\left(\frac{5}{3}\right) = \frac{3}{4}(-1) + c$$
$$c = \frac{29}{12}$$

$$\therefore y = \frac{3}{4}x + \frac{29}{12}$$

ANALYTICAL GEOMETRY

Converting gradient (m) into angle of inclination (θ)

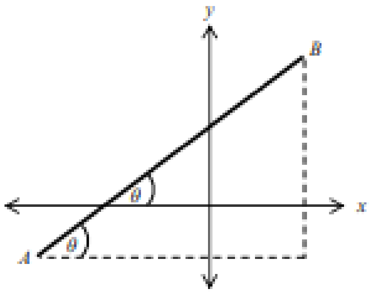
$m_{AB} = \frac{\Delta y}{\Delta x}$

and

$\tan\theta = \frac{o}{a} = \frac{\Delta y}{\Delta x}$

therefore;

$\therefore m_{AB} = \tan\theta$

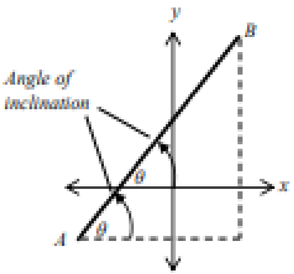


The angle of inclination (θ) is always in relation to a horizontal plane in an anti-clockwise direction.

Positive gradient:

$m > 0$
$\tan^{-1}(m) = \theta$

The reference angle is equal to the angle of inclination.

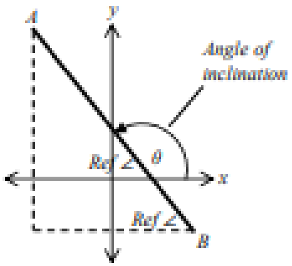


Negative gradient:

$m < 0$
$\tan^{-1}(m) = \text{ref} \angle$

Angle of inclination:
$\theta + \text{ref} \angle = 180°$ (∠'s on str. line)

The angle of inclination must be calculated from the reference angle.



Converting a positive gradient into an angle

$m > 0$
$\tan^{-1}(m) = \theta$

The reference angle is equal to the angle of inclination.

Given: $A(-1; -6)$ and $B(3; 5)$ are two points on a straight line. Determine the angle of inclination.

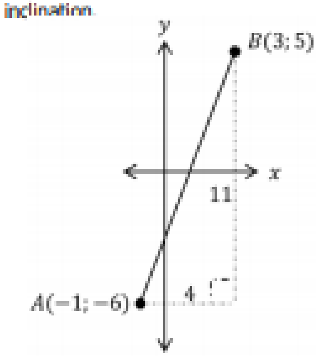
$m = \tan\theta$

$\frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$

$\frac{5-(-6)}{3-(-1)} = \tan\theta$

$\tan^{-1}\left(\frac{11}{4}\right) = \theta$

$\therefore \theta = 70°$

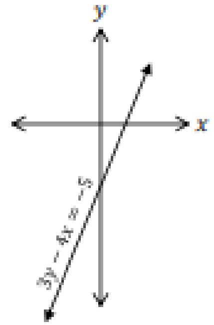


EXAMPLE

Given: straight line with the equation $3y - 4x = -5$. Determine the angle of inclination correct to two decimal places.

$3y - 4x = -5$ - make y the subject
$3y = 4x - 5$
$y = \frac{4}{3}x - \frac{5}{3}$ - note that m>0

$m = \tan\theta$ - sub. m and solve θ
$\frac{4}{3} = \tan\theta$
$\tan^{-1}\left(\frac{4}{3}\right) = \theta$
$\therefore \theta = 53{,}13°$ - m>0; ref. ∠ = angle of inclination

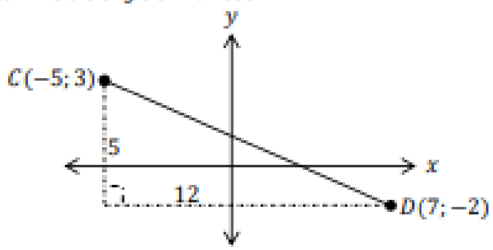


Converting a negative gradient into an angle

$m < 0$
$\tan^{-1}(m) = \text{ref} \angle$

Angle of inclination:
$\theta + \text{ref} \angle = 180°$ (∠'s on str. line)

Given: $C(-5; 3)$ and $D(7; -2)$ are two points on a straight line. Determine the angle of inclination.



$m = \tan\theta$

$\frac{5}{12} = \tan\theta$

$\tan^{-1}\left(\frac{5}{12}\right) = \theta$

$\therefore \text{ref}. \angle = 22{,}6°$

$\theta + \text{ref}. \angle = 180°$
$\theta = 180° - 22{,}6°$
$= 157{,}4°$

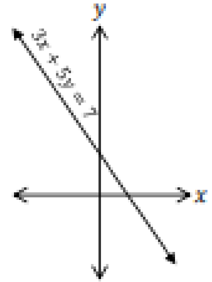
EXAMPLE

Given: straight line with the equation $3x + 5y = 7$. Determine the angle of inclination correct to two decimal places.

$3x + 5y = 7$ - make y the subject
$5y = -3x + 7$
$y = -\frac{3}{5}x - \frac{7}{5}$ - note that m<0

$m = \tan\theta$ - sub. m as a positive value to determine the ref. ∠
$\frac{3}{5} = \tan\theta$
$\tan^{-1}\left(\frac{3}{5}\right) = \theta$
$\therefore \text{ref}. \angle = 30{,}96°$

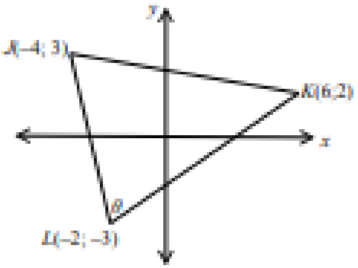
$\theta + \text{ref} \angle = 180°$ - m<0; ref. ∠ + θ = 180°
$\theta = 180° - 30{,}96°$
$\theta = 149{,}04°$

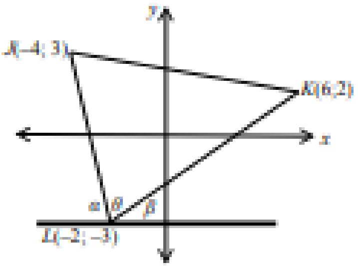


Analytical Geometry

Finding an angle that is not in relation to a horizontal plane

Construct a horizontal plane, parallel to the x-axis. This will allow you to use the 'sum of adjacent angles on a straight line' in order to calculate the value of the angle.





$m_{JL} = -\frac{6}{2} = -3$

$m = tan\,\alpha$

$3 = tan\,\alpha$

$tan^{-1}(3) = \alpha$

$71,6^\circ = \alpha$

$m_{KL} = \frac{5}{8}$

$m = tan\,\beta$

$\frac{5}{8} = tan\,\beta$

$tan^{-1}\left(\frac{5}{8}\right) = \beta$

$32^\circ = \beta$

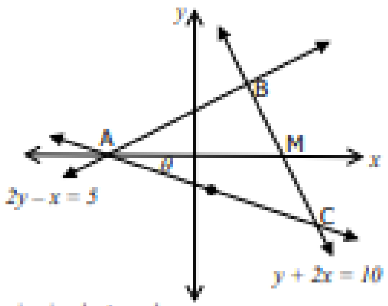
$\theta = 180^\circ - (\alpha + \beta)$

$= 180^\circ - (71,6^\circ + 32^\circ)$

$= 76,4^\circ$

EXAMPLE

Given: In the diagram: Straight line with the equation $2y - x = 5$, which passes through A and B. Straight line with the equation $y + 2x = 10$, which passes through B and C. M is the midpoint of BC. A, B and C are vertices of ΔABC. $M\hat{A}C = \theta$. A and M lie on the x-axis.



Questions:

1. Determine the following:
   a. The co-ordinates of A
   b. The co-ordinates of M.
   c. The co-ordinates of B.
2. What type of triangle is ABC? Give a reason for your answer.
3. If $A(-5;0)$ and $B(3;4)$, show that $AB = BC$ (leave your answer in simplest surd form).
4. If $C(7;-4)$, determine the co-ordinate of N, the midpoint of AC.
5. Hence, or otherwise, determine the length of MN.
6. If $ABCD$ is a square, determine the co-ordinates of D.
7. Solve for θ correct to one decimal places.

Solutions:

a. $2y - x = 5$

$2y = x + 5$

$y = \frac{1}{2}x + \frac{5}{2}$

x – cut : $0 = \frac{1}{2}x + \frac{5}{2}$

$0 = x + 5$

$-5 = x$

$\therefore A(-5;0)$

b. $y + 2x + 10$

$y = -2x + 10$

x – cut : $0 = -2x + 10$

$2x = 10$

$x = 5$

$\therefore M(5;0)$

c. $\frac{1}{2}x + \frac{5}{2} = -2x + 10$

$x + 5 = -4x + 20$

$5x = 15$

$x = 3$

$y = -2(3) + 10$

$y = 4$

$\therefore B(3;4)$

2. ABC is a right-angled triangle:

$m_{AB} \times m_{BC} = -1$

$\therefore \hat{b} = 90^\circ$

3. $d_{AB} = \sqrt{(-5-3)^2 + (0-4)^2}$

$= 4\sqrt{5}$

$d_{BC} = \sqrt{(3-7)^2 + (4-(-4))^2}$

$= 4\sqrt{5}$

$\therefore AB = BC$

4. $N(x;y) = \left(\frac{-5+7}{2};\frac{0+(-4)}{2}\right)$

$N(1;-2)$

5. $MN = 2\sqrt{5}$ (Midpt theorem)

6. If $ABCD$ is a square, then AC is the diagonal, which makes N the midpoint for both diagonals $\therefore D(-3;-8)$

7. $m_{AC} = \frac{\Delta y}{\Delta x}$

$= \frac{0-(-4)}{-5-7}$

$= -\frac{1}{3}$

$m = tan\,\theta$

$-\frac{1}{3} = tan\,\theta$

$tan^{-1}\left(-\frac{1}{3}\right) = \theta$

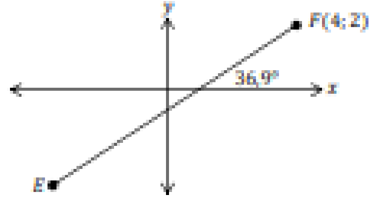
$\theta = 18,4^\circ$

Converting an angle into a gradient

Sub. the ref. ∠ into $m = \tan\theta$.

Remember to add the – sign to answers for negative gradients.

Given: E and $F(4;2)$ are points on a straight line with an angle of inclination of 36,9°. Determine the value of m correct to two decimal places.



$m = \tan\theta$

$m = \tan(36,9^\circ)$

$m = 0,75$

HELPFUL HINTS:

1. Make a quick rough sketch if you are given co-ordinates without a drawing.
2. Always make y the subject if you are given straight line equations.
3. Know your types of triangles and quadrilaterals. Proving them or using their properties is a common occurrence.
4. The angle of inclination is ALWAYS in relation to the horizontal plane.