

NOTES DIFFERENTIAL CALCULUS

THE CONCEPT OF A LIMIT

EXAMPLE 1

Determine the following limits:

$$\begin{aligned} \text{(a)} \quad \lim_{x \rightarrow 1} (2x^2 + 4) \\ &= (2(1)^2 + 4) \\ &= 6 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow 2} \frac{x^2 - 4}{x + 1} \\ &= \frac{(2)^2 - 4}{2 + 1} \\ &= 0 \end{aligned}$$

$$\text{(c)} \quad \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

In this example, it is first necessary to simplify the expression before determining the limit, as the denominator will be zero if $x = 1$ and division by zero is undefined.

$$\begin{aligned} \therefore \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{(x + 1)(x - 1)}{(x - 1)} \\ &= \lim_{x \rightarrow 1} (x + 1) \\ &= 1 + 1 = 2 \end{aligned}$$

Remember the following factorisation principles:

$$a^2 - b^2 = (a + b)(a - b)$$

$$(b - a) = -(a - b)$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

RULES OF DIFFERENTIATION

The following rules of differentiation are applicable for determining the gradient of a function given that the value of k is a constant.

Rule 1 If $f(x) = kx^n$, then $f'(x) = knx^{n-1}$

Example:

If $f(x) = 8x^3$, then $f'(x) = 8 \times 3x^{3-1} = 24x^2$

Rule 2 If $f(x) = kx$, then $f'(x) = k$

Example:

If $f(x) = 9x$, then $f'(x) = 9$

Rule 3 If $f(x) = k$, then $f'(x) = 0$

Example:

If $f(x) = 9$, then $f'(x) = 0$

If $f(x) = p$ (where p is a constant), then $f'(x) = 0$

Rule 1: Multiply out before you differentiate.

EXAMPLE 11

Determine the following:

(a) $f'(x)$ if $f(x) = (4x - 3)^2$

Here we first need to expand and simplify so as to remove the brackets.

$$f(x) = (4x - 3)^2$$

$$\therefore f(x) = 16x^2 - 24x + 9$$

$$\therefore f'(x) = 16 \times 2x^{2-1} - 24 + 0$$

$$\therefore f'(x) = 32x - 24$$

Rule 2: To get rid of the monomial in denominator ,write each term over the denominator and simplify:

$$\begin{aligned} \text{(c)} \quad & D_x \left[\frac{x^2 - x - 6}{2x^2} \right] \\ &= D_x \left[\frac{x^2}{2x^2} - \frac{x}{2x^2} - \frac{6}{2x^2} \right] \\ &= D_x \left[\frac{1}{2} - \frac{1}{2x} - \frac{3}{x^2} \right] \\ &= D_x \left[\frac{1}{2} - \frac{1}{2}x^{-1} - 3x^{-2} \right] \\ &= 0 - \frac{1}{2}(-1)x^{-1-1} - 3(-2)x^{-2-1} \\ &= \frac{1}{2}x^{-2} + 6x^{-3} \\ &= \frac{1}{2x^2} + \frac{6}{x^3} \end{aligned}$$

Rule 3: To get rid of the binomial in the denominator, factorise, simplify then differentiate.

$$\begin{aligned} \text{(d)} \quad & D_x \left[\frac{3x^3 - 7x^2 - 6x}{3 - x} \right] \\ &= D_x \left[\frac{x(3x^2 - 7x - 6)}{3 - x} \right] \\ &= D_x \left[\frac{x(3x + 2)(x - 3)}{3 - x} \right] \\ &= D_x \left[\frac{x(3x + 2)(x - 3)}{-(x - 3)} \right] \\ &= D_x \left[-x(3x + 2) \right] \\ &= D_x \left[-3x^2 - 2x \right] \\ &= -6x - 2 \end{aligned}$$

RULE 4: Get rid of square and cube roots

$$(b) \quad \frac{dy}{dx} \text{ if } y = \frac{1}{2\sqrt[4]{x^3}}$$

Here we need to first get rid of the divide line over the variable as well as the root sign.

$$y = \frac{1}{2\sqrt[4]{x^3}}$$

$$\therefore y = \frac{1}{2x^{\frac{3}{4}}} \quad (\sqrt[n]{a^m} = a^{\frac{m}{n}})$$

$$\therefore y = \frac{1}{2}x^{-\frac{3}{4}} \quad (\text{Exponential definition (c)})$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}x^{-\frac{3}{4}} - \frac{3}{4}x^{-\frac{3}{4}-1} \quad (\text{Rule 1})$$

$$\therefore \frac{dy}{dx} = -\frac{3}{8}x^{-\frac{7}{4}}$$

$$\therefore \frac{dy}{dx} = -\frac{3}{8x^{\frac{7}{4}}}$$

$$\therefore \frac{dy}{dx} = -\frac{3}{8\sqrt[4]{x^7}}$$

Determine:

(a) $f'(x)$ if $f(x) = 3x^6$	(a) $\frac{dy}{dx}$ if $y = 3x^6$	(a) $D_x[3x^6]$
Rule 1: $f(x) = 3x^6$ $\therefore f'(x) = 3 \times 6x^{6-1}$ $\therefore f'(x) = 18x^5$	Rule 1: $y = 3x^6$ $\therefore \frac{dy}{dx} = 3 \times 6x^{6-1}$ $\therefore \frac{dy}{dx} = 18x^5$	Rule 1: $D_x[3x^6]$ $= 3 \times 6x^{6-1}$ $= 18x^5$
(b) $f'(x)$ if $f(x) = 10x$	(b) $\frac{dy}{dx}$ if $y = 10x$	(b) $D_x[10x]$
Rule 2: $f(x) = 10x$ $\therefore f'(x) = 10$	Rule 2: $y = 10x$ $\therefore \frac{dy}{dx} = 10$	Rule 2: $D_x[10x]$ $= 10$
(c) $f'(x)$ if $f(x) = -8$	(c) $\frac{dy}{dx}$ if $y = -8$	(c) $D_x[-8]$
Rule 3: $f(x) = -8$ $\therefore f'(x) = 0$	Rule 3: $y = -8$ $\therefore \frac{dy}{dx} = 0$	Rule 3: $D_x[-8]$ $= 0$
(d) $f'(x)$ if $f(x) = m$ (m is a constant)	(d) $\frac{dy}{dx}$ if $y = m$ (m is a constant)	(d) $D_x[m]$ (m is a constant)
Rule 3: $f(x) = m$ $\therefore f'(x) = 0$	Rule 3: $y = m$ $\therefore \frac{dy}{dx} = 0$	Rule 3: $D_x[m]$ $= 0$
(e) $g'(x)$ if $g(x) = \frac{x^3}{3}$	(e) $\frac{dy}{dx}$ if $y = \frac{x^3}{3}$	(e) $D_x\left[\frac{x^3}{3}\right]$
$g(x) = \frac{x^3}{3}$ First use exp def (a): $\therefore g(x) = \frac{1}{3}x^3$ Now use Rule 1: $\therefore g'(x) = \frac{1}{3} \times 3x^{3-1}$ $\therefore g'(x) = x^2$	$y = \frac{x^3}{3}$ First use exp def (a): $\therefore y = \frac{1}{3}x^3$ Now use Rule 1: $\therefore \frac{dy}{dx} = \frac{1}{3} \times 3x^{3-1}$ $\therefore \frac{dy}{dx} = x^2$	$D_x\left[\frac{x^3}{3}\right]$ $= D_x\left[\frac{1}{3}x^3\right]$ exp def (a) $= \frac{1}{3} \times 3x^{3-1}$ Rule 1 $= x^2$

EXAMPLE 5



Find, from first principles, the gradient of $f(x) = 3x$ at any point.

Solution

We know that the graph of $y = 3x$ is a straight line with a gradient of 3.
By using first principles, we can verify this fact.

Step 1 Write down the formula for finding gradient from first principles:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Step 2 Write down the given $f(x)$ and then determine $f(x+h)$:

$$\begin{aligned} f(x) &= 3x & \therefore f(x+h) &= 3(x+h) \\ & & \therefore f(x+h) &= 3x + 3h \end{aligned}$$

Step 3 Substitute the expressions for $f(x)$ and $f(x+h)$ into the formula and then simplify the expression and evaluate the limit:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{3x + 3h - 3x}{h} \\ \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{3h}{h} \\ \therefore f'(x) &= \lim_{h \rightarrow 0} 3 \\ \therefore f'(x) &= 3 \end{aligned}$$

Hence the gradient of the line is 3.

EXAMPLE 7

- Find, from first principles, the gradient of $f(x) = 1 - 3x^2$ at any point.
- Hence find $f'(-4)$, the derivative of f at $x = -4$.
- What is the gradient of the tangent to f at $x = 5$?

Solution

(a) **Step 1**

Write down the formula for finding gradient from first principles:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Step 2

Write down the given $f(x)$ and then determine $f(x+h)$:

$$f(x) = 1 - 3x^2$$

$$\therefore f(x+h) = 1 - 3(x+h)^2$$

$$\therefore f(x+h) = 1 - 3(x^2 + 2xh + h^2)$$

$$\therefore f(x+h) = 1 - 3x^2 - 6xh - 3h^2$$

Step 3

Substitute the expressions for $f(x)$ and $f(x+h)$ into the formula and then simplify the expression and evaluate the limit:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore f'(x) = \lim_{h \rightarrow 0} \frac{(1 - 3x^2 - 6xh - 3h^2) - (1 - 3x^2)}{h}$$

$$\therefore f'(x) = \lim_{h \rightarrow 0} \frac{1 - 3x^2 - 6xh - 3h^2 - 1 + 3x^2}{h}$$

$$\therefore f'(x) = \lim_{h \rightarrow 0} \frac{-6xh - 3h^2}{h}$$

$$\therefore f'(x) = \lim_{h \rightarrow 0} \frac{h(-6x - 3h)}{h}$$

$$\therefore f'(x) = \lim_{h \rightarrow 0} (-6x - 3h)$$

$$\therefore f'(x) = -6x - 3(0)$$

$$\therefore f'(x) = -6x$$

- (b) Since $f'(x) = -6x$ represents the gradient of the graph at any point on the graph, it is now easy to determine the gradient (derivative) at $x = -4$:

$$f'(x) = -6x$$

$$\therefore f'(-4) = -6(-4) = 24$$

- (c) The gradient of the tangent to the graph at $x = 5$ can now also be determined:

$$f'(x) = -6x$$

$$\therefore f'(5) = -6(5) = -30$$